Strong Metric Dimension of Infinite Graphs

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Overview

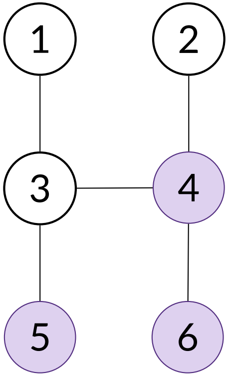
- Metric dimension
- Strong metric dimension
- Non-locally finite graphs
- Motivating example
- Results and conjectures

Motivation for metric dimension

How many sensors do you need in a network to uniquely identify another node based on its distance from each sensor? This question is answered by **metric dimension** and its variants such as truncated metric dimension and threshold dimension.

Resolving set

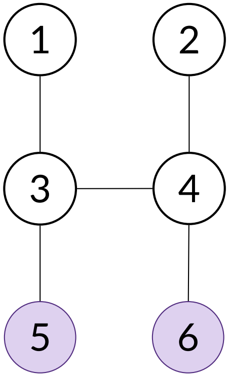
A **resolving set** is a set $R \subseteq V$ such that, for all distinct $u, v \in V$, there exists a vertex $r \in R$ such that $d(r, u) \neq d(r, v)$. Also known as a metric basis.



An example of a resolving set.

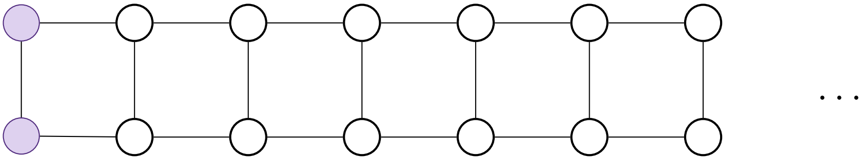
Metric dimension

The **metric dimension** of a graph G, written $\beta(G)$, is the smallest cardinality of all resolving sets on G.



This is a minimal resolving set, so the metric dimension is two.

Metric dimension of infinite graphs



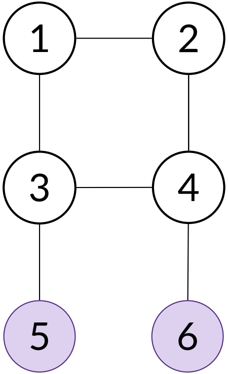
The infinite ladder has metric dimension two.

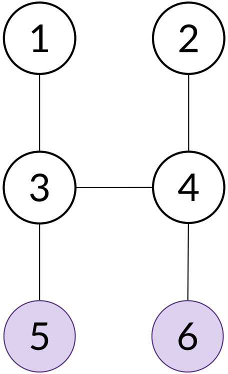
Background

Although there is lots of research on metric dimension, it has not been applied to infinite graphs as much.
A previous paper demonstrated that adding vertices can make the metric dimension of an infinite graph oscillate between finite and infinite.

Motivation for a stronger invariant

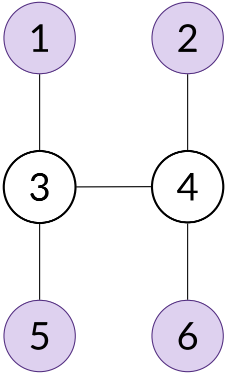
Sebő and Tannier showed that it is possible to have two non-isomorphic graphs with the same resolving sets and distance vectors.





Strong resolving set

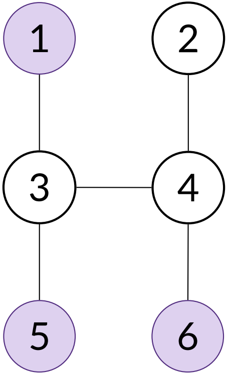
A **strong resolving set** is a set $R \subseteq V$ such that, for all distinct $u, v \in V$, there exists a vertex $r \in R$ such that a shortest r–u path contains v or a shortest r–v path contains u.



A strong resolving set.

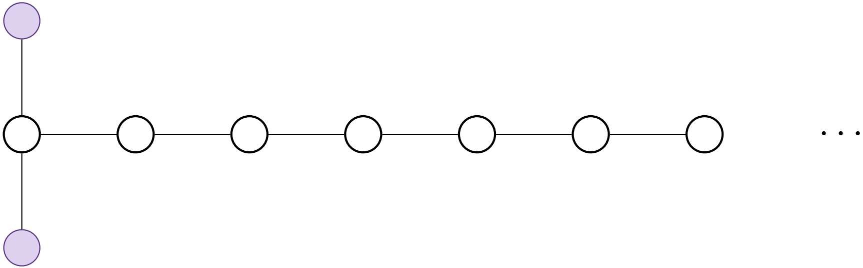
Strong metric dimension

The **strong metric dimension** of a graph G, written $\beta_S(G)$, is the smallest cardinality of all strong resolving sets on G.



A minimal strong resolving set.

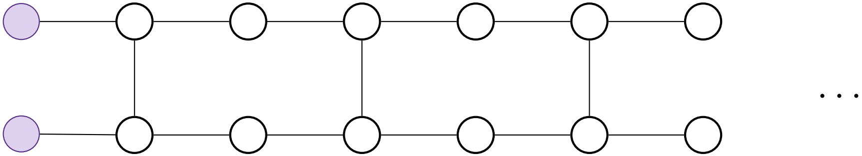
Strong metric dimension of infinite graphs



An example of an infinite graph with finite strongly resolving set.

Motivating example

The broken ladder, although a subgraph of the infinite ladder, has infinite strong metric dimension.

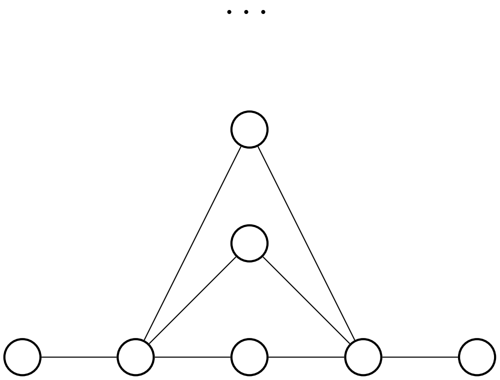


The broken ladder, with highlighted resolving set and infinite strong resolving set.

Degree and local finite-ness

The **degree** of a vertex is the number of neighboring vertices. A **locally finite** graph is one where each vertex has finite degree.

Example



A graph that has two vertices of infinite degree.

Non-locally finite graphs

Cáceres et al. showed that for an infinite graph G, if $\beta(G) = k$, then G has maximum degree $3^k - 1$. By the contrapositive, if $\beta(G) < \infty$, then G is locally finite.

Lemma

Let G be an infinite but locally finite graph and let $W \subseteq V(G)$ be a finite set of vertices. Let $R = \{v_1, v_2, \ldots\}$ be some infinite subset of vertices. Then, $\lim_{i\to\infty} d(v_i, W) = \infty$.

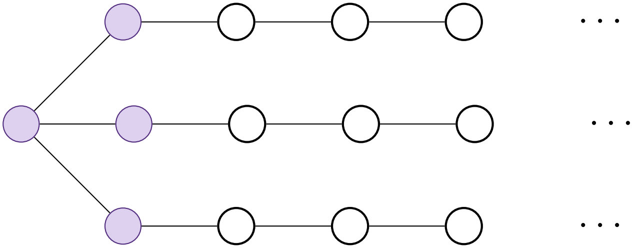
Proof of lemma

If this sequence did not tend to infinity, some k must appear infinitely many times. Then, the k-th neighborhood is not finite, which is a contradiction.

Main result

Let G be an infinite but locally finite graph. If there exist sets $R_1, R_2 \subseteq V(G)$ such that R_1 and R_2 are infinite but can be disconnected by removing a finite set of vertices, then $\beta_S(G) = \infty$.

Example



Applying the theorem with the disconnecting set being the highlighted vertices.

Outline of proof

- Assume a finite strongly resolving set W_0 and finite disconnecting set W_1.

- $W := W_0 \cup W_1$ still has to be a resolving set.

- Let $u \in R_1, v \in R_2$ be "far enough" away from each other.

Outline of proof

- Assume $w \in W$ resolves u and v. Without loss of generality, v is on the shortest u–w path.

- The shortest u–v path contains $w_0 \in W$ separating R_1 and R_2, so $d(u, v) = d(u, w_0) + d(v, w_0) \geq d(u, W) + d(v, W)$.

- Let $w' \in W$ such that $d(u, w') = d(u, W)$.

- $d(u, w) \leq d(u, w') + d(w, w') \leq d(u, v)$, a contradiction!

Corollary

A **end** is an equivalence class of rays where two rays are equivalent if they can’t be disconnected by removing finitely many edges.

Corollary

If a graph has more than one end, then it has infinite strong metric dimension.

Rays

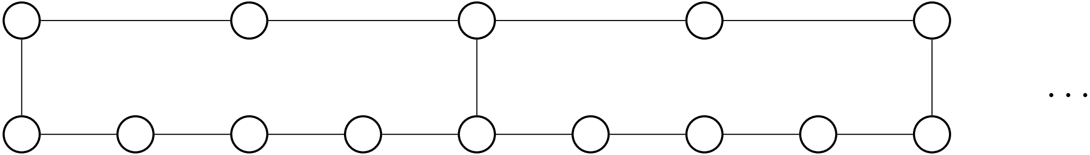
A **ray** is an infinite graph where $V = \{v_1, v_2, v_3, \ldots\}$ and $E = \{v_1v_2, v_2v_3, \ldots\}$.
A **shortest ray** is a ray $R \subseteq G$ such that for any $u, v \in R$, the path u–v in R is a shortest path in G.

Conjecture

- For an infinite graph G and a finite subset of vertices W, let $\vec{G}_W$ be a partially directed graph where edges directed according to the direction of shortest rays starting in W.

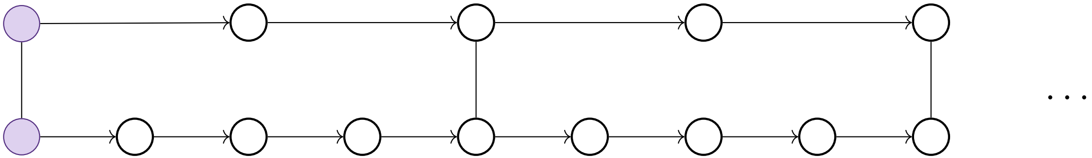
- If there is a pair $u, v \in \vec{G}_W$ for every finite choice of W such that there is no u–v path and there is no v–u path, then $\beta_S(G) = \infty$.

Example



An infinite graph G with infinite strong metric dimension.

Example



The corresponding digraph $\vec{G}_W$, where W consists of highlighted vertices.

Future work

We are working towards a characterization of graphs that have infinite strong metric dimension but finite metric dimension by proving the converse of our result.
Similar characterizations might be possible using other graph invariants like truncated metric dimension.

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