Strong Metric Dimension of Infinite Graphs

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Graph theory

A **graph** is defined as a set of vertices and a set of edges that define which vertices are connected. More precisely, A graph is an ordered pair $G = (V, E)$ such that V is a set and $E \subseteq \{\{x, y\} : x, y \in V, x \neq y\}$. In other words, a graph is a set of nodes and the set of edges that connect pairs of nodes.

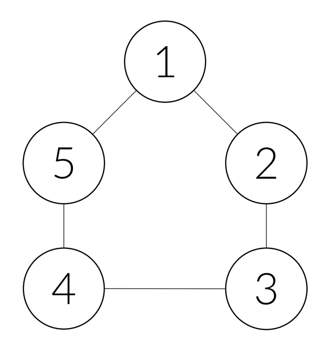

Figure 1. A graph with vertex set $V = \{1, 2, 3, 4, 5\}$.

Graphs are commonly used to model relationships in biology, physics, chemistry, social science, and computer science.

Metric dimension

A **resolving set** is a set $R \subseteq V$ such that, for all distinct $u, v \in V$, there exists a vertex $r \in R$ such that $d(r, u) \neq d(r, v)$. The **metric dimension** of a graph $G = (V, E)$, denoted $\beta(G)$, is defined to be the smallest cardinality of resolving sets on G.

In other words, the metric dimension of a graph is the minimum number of vertices needed to identify each vertex in the graph based solely on the shortest path distances to the resolving vertices.

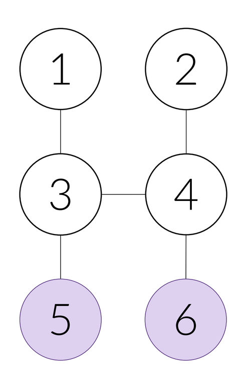

Figure 2. A graph with resolving set $W = \{5, 6\}$. [?]

Strong metric dimension

A **strongly resolving set** is a set $R \subseteq V$ such that, for all distinct $u, v \in V$, there exists a vertex $r \in R$ such that either u is on a shortest r–v path or v is on a shortest r–u path. The **strong metric dimension** of a graph $G = (V, E)$, denoted $\beta_S(G)$, is the smallest cardinality of strongly resolving sets on G.

Two non-isomorphic graphs with the same vertex set can have the same resolving set. However, two graphs only have the same strongly resolving set if they are isomorphic.

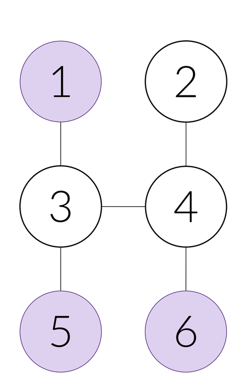

Figure 3. The same graph with strongly resolving set $W = \{1, 5, 6\}$.

Proposition. For any graph G, $\beta_S(G) \geq \beta(G)$.

Abstract

We investigate the strong metric dimension of graphs with infinitely many vertices. Our main result shows that if a graph has two infinite sets of vertices that can be disconnected by removing a finite set of vertices, then it must have infinite strong metric dimension.

Non-locally finite graphs

The **degree** of a vertex in a graph is the number of neighboring vertices. A **locally finite** graph is one where each vertex has finite degree.

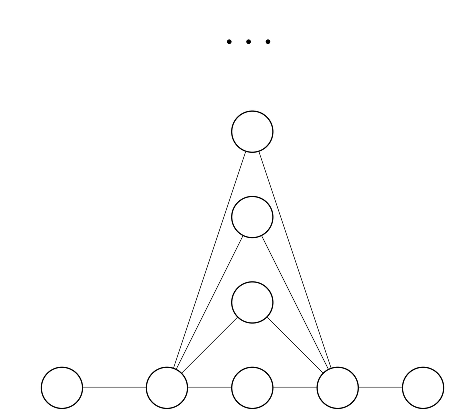

Figure 4. A graph that is not locally finite.

Our theorem specifically refers to graphs that are locally finite since it has already been proven that a non-locally finite graph must have infinite metric dimension, and thus infinite strong metric dimension.

Theorem. Let G be a graph. If $\beta(G) = k$, then G has maximum degree at most $3^k - 1$.

This theorem was proved by Chartrand, Poisson, and Zhang for finite graphs [?] and by Cáceres et al. for infinite graphs [?].

Corollary. If $\beta(G) < \infty$, then G is locally finite.

Corollary. If G is not locally finite, then $\beta_S(G) = \infty$.

Theorem

Lemma. Let G be a locally finite graph and let $W \subseteq V(G)$ be a finite set of vertices. Let $R = \{v_1, v_2, \ldots\} \subseteq V(G)$. Then, $\lim_{i\to\infty} d(v_i, W) = \infty$.

Proof. Suppose, for a contradiction, that the sequence does not tend to infinity and that G is locally finite. Then, there must be an element that appears infinitely many times. Let k be the smallest such number. Since k occurs infinitely many times, the k-th neighborhood is not finite, contradicting the assumption that G is locally finite. □

Theorem. Let G be a locally finite graph. If there exist sets $R_1, R_2 \subseteq V(G)$ such that R_1 and R_2 are infinite but can be disconnected by removing a finite set of vertices, then $\beta_S(G) = \infty$.

Proof. Assume, for a contradiction, that W_0 is a finite strongly resolving set and let W_1 be the finite disconnecting set. Then, $W = W_0 \cup W_1$ is clearly still a strongly resolving set, as all vertices are strongly resolved by at least one vertex in W_1 and hence at least once vertex in W.

Let $m = \max\{d(w, w') : w, w' \in W\}$. By the lemma, there exist vertices $u \in R_1, v \in R_2$ such that $d(u), d(v) > m$. We claim that W does not resolve u and v. Assume, for a contradiction, that w does resolve u and v. Without loss of generality, assume the shortest u–w path contains v.

First, $d(u, v) \geq d(u) + d(v)$, since the shortest u–v path P contains a vertex $w_0 \in W$. Then, $d(u, w) \leq d(u, v)$, which contradicts the assumption that v is on a shortest u–w path.

□

Example

In the below graph, let R_1 be the top ray and R_2 be the union of the upper and lower rays. Then, applying the theorem using the vertex connecting all the rays and the first vertex of each ray shows that this graph has infinite strong metric dimension.

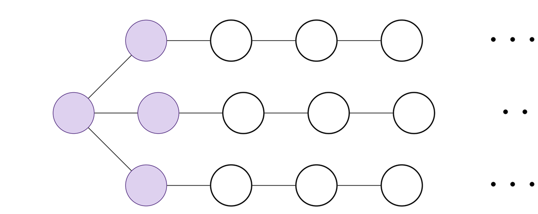

Figure 5. Applying the theorem with W being the highlighted vertices.

The broken ladder

The **infinite ladder** is an infinite graph with vertex set $V = \{v_{i,j} : i \in \mathbb{N}, j \in \{0, 1\}\}$ and edge set

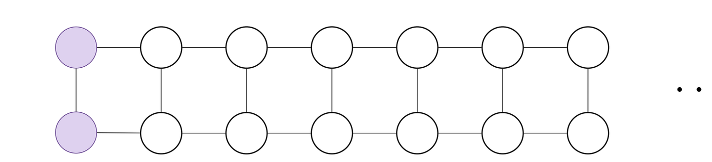

Figure 6. The infinite ladder.

Proposition. The infinite ladder has metric dimension 2 and strong metric dimension 2 with the resolving set $W = \{v_{1,0}, v_{1,1}\}$.

The **broken ladder** is a subgraph of the infinite ladder where every second rung is removed. This was a motivating example for our investigation, since it has finite metric dimension and infinite strong metric dimension.

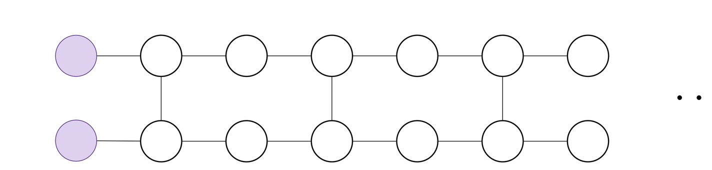

Figure 7. The broken ladder.

Proposition. The broken ladder has metric dimension 2 with the same resolving set and infinite strong metric dimension.

Remark. Note that, although the broken ladder has infinite strong metric dimension, there are no two infinite sets satisfying the statement of the theorem. The obvious choice, the upper and lower rays, cannot be disconnected by removing a finite number of edges. This means that there is a missing condition to prove the converse of the theorem.

Furthermore, we hope to characterize graphs that have finite metric dimension and infinite strong metric dimension, such as the broken ladder. Similar characterizations of infinite graphs may also be possible using concepts related to strong metric dimension, such as truncated metric dimension and threshold dimension.

References